ON A CLASS OF DIVERSE MARKET MODELS

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Abstract. A market model in Stochastic Portfolio Theory is a finite system of strictly positive stochastic processes. Each process represents the capitalization of a certain stock. If at any time no stock dominates the entire market, which means that its share of total market capitalization is not very close to one, then the market is called diverse. There are several ways to outperform diverse markets and get an arbitrage opportunity, and this makes these markets interesting. A feature of real-world markets is that stocks with smaller capitalizations have larger drift coefficients. Some models, like the Volatility-Stabilized Model, try to capture this property, but they are not diverse. In an attempt to combine this feature with diversity, we construct a class of market models. We find simple, easy-to-test sufficient conditions for them to be diverse and other sufficient conditions for them not to be diverse.

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1. Introduction

Stochastic Portfolio Theory (SPT) is a recently developed area of financial mathematics. It is a flexible framework for analyzing portfolio behavior and equity market structure. See the book [Fer02] and the more recent survey [FK09] for detailed treatment of this topic.

Fix n, the number of stocks. Denote by $X_i(t)$ the total capitalization of ith stock at time t. Let

$$X(t) = X_1(t) + \ldots + X_n(t) \quad \text{and} \quad \mu_i(t) = \frac{X_i(t)}{X(t)}, \quad i = 1, \ldots, n,$$

be the total capitalization of the market at time t and the *market weights* of each stock, respectively. Fix a threshold $\delta \in (0, 1)$. The market is called *δ-diverse* if for every $t \geq 0$ and $i = 1, \ldots, n$ we have:

$$\mu_i(t) < 1 - \delta.$$

This definition was introduced in [Fer02]. Intuitively it means that, at any given moment, no stock dominates almost the entire market.

A central problem in SPT is to construct a portfolio which outperforms the market, or, speaking in terms of SPT, *allows an arbitrage relative to the market.* A few portfolios which outperform the market were constructed in the articles [FKK05], [FK05] and the survey [FK09]. In these articles, it is proved that diverse markets can be outperformed. This makes diverse market models interesting.

A few market models have recently attracted considerable attention. First, let us mention Geometric Brownian Motions with Rank-Dependent Drifts (see [BFK05], [PP08] and [CP10]) and its generalization, the hybrid Atlas models (see [IPB+11]). However, they are not diverse.

In real-world markets stocks with lower market weights have larger drift and larger volatility. The Volatility-Stabilized Model, introduced in [FK05] and further elaborated in [Pal11], attempt to capture this effect; see also the preprint [Pic13] for some generalizations. Unfortunately, it is also not diverse.

In this paper, we desire to combine diversity with this property of stocks with lower market weights. We introduce the following general class of models. Assume $g : (0, 1-\delta) \to \mathbb{R}$ is a continuous locally Lipschitz function such that $\lim_{s\to(1-\delta)-} g(s) = +\infty$. Let $W = ((W_1(t), \ldots, W_n(t)), t \geq$

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0) be an n-dimensional Brownian motion. The equations defining our market model are:

$$d\log X_i(t) = -g(\mu_i(t))dt + dW_i(t),\quad i = 1, \ldots, n.$$

We find conditions on the function g which guarantee that the market is δ-diverse, or that it is not δ-diverse.

We also refer the reader to the following articles on this topic. Some diverse models were introduced in the article [FK05] and were also mentioned in the survey [FK09, Chapter 7]. The paper [OR06] gives a measure-change method for constructing diverse market models.

First, we study the case $n = 2$ (two stocks). The proof in this case is easier than in the general case, and we are able to obtain necessary and sufficient conditions for diversity. Then, we consider the general case $n \ge 2$. Here, we are only able to find some necessary conditions for diversity and some other sufficient conditions for it. Our main technique is Feller's test for explosions, taken from [Dur96, Section 6.2].

The paper is organized as follows. In Section 2, we present basic definitions and statements of theorems. In Section 3, we provide proofs for the case $n = 2$, and in Section 4, we consider the general case $n \ge 2$. The Appendix contains the statements and proofs of some auxillary lemmas.

2. Main Results

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbf{P})$, where the filtration $(\mathcal{F}_t)_{t\ge0}$ satisfies the usual conditions: namely, it is right-continuous and augmented by $\mathbf{P}$-null sets. We denote the $n \times n$-identity matrix by I_n. For an open interval $I \subseteq \mathbb{R}$, a function $f : I \to \mathbb{R}$ is called *locally Lipschitz* if for every $[a, b] \subseteq I$ there exists a constant $K > 0$ such that for every $x, y \in [a, b]$ we have: $|f(x) - f(y)| \le K|x - y|$.

Fix $d \ge 1$. Let

$$W = (W_1(t), \ldots, W_d(t), t \ge 0)$$

be a d-dimensional Brownian motion. Assume the filtration $(\mathcal{F}_t)_{t\ge0}$ is generated by this Brownian motion. Denote by n the number of stocks. Assume

$$\gamma = (\gamma_1(t), \ldots, \gamma_n(t), t \ge 0)$$

is an $\mathbb{R}^n$-valued progressively measurable stochastic process, and

$$\sigma = (\sigma(t) = (\sigma_{ij}(t))_{1\le i\le n, 1\le j\le d}, t \ge 0)$$

is a matrix-valued $(\mathcal{F}_t)_{t\ge0}$-progressively measurable stochastic process. (The size of the matrix is $n \times d$.) We impose a technical condition: for every finite $T \ge 0$, each γ_i is integrable and each σ_{ij} is square-integrable on $[0, T]$ a.s.

Definition 1. Assume that an n-dimensional process

$$\mathcal{M} = (X_1(t), \ldots, X_n(t), t \ge 0)$$

taking values in $(0, \infty)^n$ satisfies the following system of equations:

$$d\log X_i(t) = \gamma_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t),\quad i = 1, \ldots, n.$$

Then $\mathcal{M}$ is called the *market model of* n *stocks* with *growth rates* γ_i and *volatility matrix* σ. The value $X_i(t)$ is called the *capitalization of* i*th stock at time* $t \ge 0$.

We have already defined market weights and the notion of diversity in the introduction. Loosely speaking, a market is diverse if at every moment no stock dominates almost the entire market. This can be viewed as a consequence of an antitrust legislation. On this, see an interesting paper [SF11]. Note the strict inequality $\mu_i(t) < 1 - \delta$ instead of $\mu_1(t) \le 1 - \delta$ in [Fer02] and [FK09]. Let us introduce a class of market models. We fix the diversity threshold $\delta \in (0, 1)$.

Definition 2. Assume a continuous function $g : (0, 1-\delta) \to \mathbb{R}$ satisfies the following conditions: $\lim_{x\to(1-\delta)-} g(x) = +\infty$, and g is locally Lipschitz. Then the function g is called *admissible*.

For every admissible g, consider the market model defined by: $n = d$, $\gamma_i(t) = -g(\mu_i(t))$ and $\sigma(t) = I_n$. In other words, consider the market model $\mathcal{M} = (X_1(t), \ldots, X_n(t), t \geq 0)$ satisfying the system of SDE:

$$d \log X_i(t) = -g(\mu_i(t))dt + dW_i(t), \quad i = 1, \ldots, n, \tag{1}$$

where $W = (W_1(t), \ldots, W_n(t), t \geq 0)$ is an n-dimensional Brownian motion.

Lemma 2.1. *Assume the market model is governed by* (1) *for some admissible function g. Let us introduce some new notation:*

$$\psi(s) := s(-g(s) + 1/2) - s^2, \quad s \in (0, 1-\delta).$$

Then the market weights satisfy the following system of equations: for $i = 1, \ldots, n$,

$$d\mu_i(t) = \left(\psi(\mu_i(t)) - \mu_i(t)\sum_{j=1}^n \psi(\mu_j(t))\right)dt + \sum_{j=1}^n (\delta_{ij}\mu_i(t) - \mu_i(t)\mu_j(t))dW_j(t), \tag{2}$$

where δ_{ij} stands for the Kronecker delta symbol.

The fairly straightforward proof (by Itô's lemma) is postponed until the Appendix. Now we can state our main results. First, we consider the case $n = 2$ (two stocks). Here, the threshold δ must be between 0 and 1/2. Indeed, $\mu_1 + \mu_2 = 1$, so at least one of market weights μ_1 and μ_2 must be greater than or equal to 1/2.

Theorem 2.2. *Fix $\delta \in (0, 1/2)$. Assume the market model is governed by the equations* (1) *for some admissible function g. Consider the case $n = 2$ (two-stock market). For $x \in [1/2, 1-\delta)$, define the function*

$$F(x) = \int_{1/2}^x \frac{g(y)}{y(1-y)}dy.$$

The market is δ-diverse, if and only if

$$\int_{1/2}^{1-\delta} e^{F(x)}dx = +\infty. \tag{3}$$

Corollary 2.3. *Fix $\delta \in (0, 1/2)$. Assume the market model is given by* (1)*, and $n = 2$ (there are two stocks). Let $C_0 := (\delta(1-\delta))^{-1}$.*

(i) Assume that $\int_{1/2}^{1-\delta} g(z)dz = +\infty$ and for some $\varepsilon > 0$ we have:

$$\int_{1/2}^{1-\delta} \exp\left((C_0 - \varepsilon)\int_{1/2}^y g(z)dz\right)dy = \infty. \tag{4}$$

Then the market is δ-diverse.

(ii) Assume again that $\int_{1/2}^{1-\delta} g(z)dz = +\infty$, but

$$\int_{1/2}^{1-\delta} \exp\left(C_0\int_{1/2}^y g(z)dz\right)dy < \infty.$$

Then the market is not δ-diverse.

(iii) Assume now that $\int_{1/2}^{1-\delta} g(z)dz < +\infty$. Then the market is not δ-diverse.

Now, consider the general case: $n \geq 2$, the number of stocks is an arbitrary integer greater than or equal to two.

Theorem 2.4. *Fix $\delta \in (0,1)$. Assume the market model is governed by the equations* (1) *for some admissible g which satisfies*

$$-\infty < \varliminf_{x\to 0+} xg(x) \le \varlimsup_{x\to 0+} xg(x) < 0. \tag{5}$$

Fix some $x_0 \in (0, 1-\delta)$. Define

$$A_1(x) := \frac{1}{x(1-x)}, \quad \text{and} \quad A_2(x) := \frac{1+(n-1)^{-1}}{2}\frac{1}{x(1-x)}.$$

(i) If we have:

$$\int_{x_0}^{1-\delta} \exp\left(\int_{x_0}^{y} A_1(z)g(z)dz\right)dy < +\infty,$$

then the market is not δ-diverse.

(ii) If we have:

$$\int_{x_0}^{1-\delta} \exp\left(\int_{x_0}^{y} A_2(z)g(z)dz\right)dy = +\infty,$$

then the market is δ-diverse.

Corollary 2.5. *Fix $\delta \in (0,1)$ and $x_0 \in (0, 1-\delta)$. Assume the market model is governed by the equations* (1) *for some admissible g that satisfies the condition* (5)*, and let*

$$a_1 := \frac{1}{\delta(1-\delta)}, \qquad a_2 := \frac{1+(n-1)^{-1}}{2}\frac{1}{\delta(1-\delta)}.$$

(i) If $\int_{x_0}^{1-\delta} g(z)dz < \infty$ and for some $\varepsilon > 0$ we have

$$\int_{x_0}^{1-\delta} \exp\left((a_2-\varepsilon)\int_{x_0}^{y} g(z)dz\right)dy = \infty,$$

then the market is δ-diverse.

(ii) If $\int_{x_0}^{1-\delta} g(z)dz = \infty$ and

$$\int_{x_0}^{1-\delta} \exp\left(a_1\int_{x_0}^{y} g(z)dz\right)dy < \infty,$$

then the market is not δ-diverse.

(iii) If $\int_{x_0}^{1-\delta} g(z)dz < \infty$, then the market is not δ-diverse.

The proofs of the first and second theorems are given in sections 3 and 4. In both proofs, we must show that no market weight hits zero or $1-\delta$. Indeed, if a certain market weight hits zero, then the respective capitalization $X_i(t)$ becomes zero, and this is forbidden by (1); whereas, if a certain market weight hits $1-\delta$, then the market ceases to be δ-diverse.

In the case $n = 2$, we can express $\mu_2(t) = 1 - \mu_1(t)$ and write an SDE for $\mu_1(t)$. Then we apply Feller's explosion test from [Dur96, Section 6.2] for μ_1. For the general case, we could use similar results for multidimensional diffusions (possibly in the spirit of [Bha78], see also [Dur96, Section 6.6]). However, we choose a different approach: taking each market weight and estimating its drift and diffusion. In this case, the proof is also based on Feller's test, but it is somewhat more technical. There are some caveats, and we were not able to find conditions which are both necessary and sufficient. We only found a condition under which the market is δ-diverse, and another condition under which it is not δ-diverse. These two conditions are pretty close to each other, but there is still a gap between them.

Consider a couple of examples. First, let $n = 2$.

Example 1. Check these conditions for $n = 2$ for the function

$$g(y) = \frac{p}{1 - \delta - y}, \quad \text{where} \quad p > 0.$$

Then $\int_{1/2}^{1-\delta} g(y)dy = +\infty$. For $\alpha > 0$ we have:

$$\exp\left(\alpha \int_{1/2}^{x} g(y)dy\right) = \exp\left(-\alpha p \log(1 - \delta - x)\right) = (1 - \delta - x)^{-\alpha p}.$$

Therefore,

$$\int_{1/2}^{1-\delta} \exp\left(\alpha \int_{1/2}^{x} g(y)dy\right) dx < \infty \quad \text{iff} \quad \alpha p < 1.$$

If $p < \delta(1-\delta)$, then take $\alpha = (\delta(1-\delta))^{-1}$ and get: $\alpha p < 1$, so the market is not δ-diverse. If $p > \delta(1-\delta)$, then take $\alpha = (\delta(1-\delta))^{-1} - \varepsilon$ for sufficiently small $\varepsilon > 0$ and get: $\alpha p \geq 1$, so the market is δ-diverse. For $p = \delta(1-\delta)$, we must plug g directly into (3) and direct calculation shows that the market is δ-diverse.

Example 2. Check these conditions for $n = 2$, for the function

$$g(y) = \frac{p}{(1 - \delta - y)^q}, \quad \text{where} \quad p, q > 0.$$

We have just discussed the case $q = 1$. For $q < 1$, we have: $\int_{1/2}^{1-\delta} g(y)dy < \infty$, so the market is not δ-diverse. For $q > 1$, we have: $\int_{1/2}^{y} g(z)dz = \frac{p}{q(1-\delta-y)^{q-1}} - C$, where C is a certain constant. Therefore, $\int_{1/2}^{1-\delta} |g(z)|dz = +\infty$. Also, for any constant $k > 0$, we have:

$$\int_{1/2}^{1-\delta} \exp\left(k \int_{1/2}^{y} g(z)dz\right) dy = \infty. \tag{6}$$

Therefore, the market is δ-diverse.

Consider the same examples for the general case: $n \geq 2$.

Example 3. Assume an admissible function g satisfies the condition (5). Assume also that in some left neighborhood of $1 - \delta$, it is given by the formula

$$g(z) = \frac{p}{1 - \delta - z}, \quad \text{where} \quad p > 0.$$

Then, similarly to Example 1, we have:
(i) if $p < 1/a_1$, then the market is not δ-diverse;
(ii) if $p \geq 1/a_2$, then the market is δ-diverse;
(iii) if $1/a_1 \leq p < 1/a_2$, then the question remains open.

Example 4. Assume the same as in the previous example, except that, in some left nieghborhood of $1 - \delta$, we have:

$$g(z) = \frac{p}{(1 - \delta - z)^q}, \quad \text{where} \quad p, q > 0.$$

Then, similarly to Example 2, we have: for $q < 1$, this market is not δ-diverse, and for $q > 1$, it is diverse.

3. Proof of Theorem 2.2 and Corollary 2.3

Proof of Theorem 2.2. Since there are only two stocks, we have: $\mu_2(t) = 1 - \mu_1(t)$. It suffices to show that μ_1 does not hit $1 - \delta$. Indeed, these two stocks are absolutely symmetric, and the proof that μ_2 does not hit $1-\delta$ is repeated verbatim. Since both of these weights are strictly less than $1-\delta$ and they add up to 1, both of them are strictly greater than δ. Therefore, they do not hit zero.

Plug $\mu_2(t) = 1 - \mu_1(t)$ into the equation (2). The market weight μ_1 satisfies the equation:

$$d\mu_1(t) = (\psi(\mu_1(t)) - \mu_1\left[\psi(\mu_1(t)) + \psi(1-\mu_1(t))\right])\,dt + (\mu_1(t) - \mu_1^2(t))dW_1(t) - (\mu_1(t) - \mu_1^2(t))dW_2(t).$$

The process $B = B(t), t \ge 0)$, where $B(t) := (W_1(t) - W_2(t))/\sqrt{2}$, is a one-dimensional standard Brownian motion, and so μ_1 satisfies the equation

$$d\mu_1(t) = b(\mu_1(t))dt + \sigma(\mu_1(t))dB(t),$$

where the drift and diffusion coefficients b_0 and σ_0 are defined as:

$$b_0(x) := \psi(x) - x(\psi(x) + \psi(1-x)), \quad \sigma_0(x) = \sqrt{2}(x - x^2).$$

Use Feller's test for explosions, see [Dur96, Section 6.2]. We apply this test for the interval $(\delta, 1-\delta)$ of values of μ_1. For this test to be applicable, the functions b_0 and σ_0 must be continuous on this interval, and

$$a_0(x) := \sigma_0^2(x) = 2x^2(1-x)^2$$

must be strictly positive on this interval (see condition (1D) in [Dur96, p.211]). It is easy to see that these conditions are indeed satisfied.

Follow the notation from [Dur96, Section 6.2]. The natural scale is defined as

$$\varphi(x) = \int_{1/2}^x \exp\left(\int_{1/2}^y -\frac{2b_0(z)}{a_0(z)}dz\right)dy.$$

Let $m(x) := 1/(\varphi'(x)a(x))$. The following conditions are equivalent (see Theorem 2.1 from [Dur96, Section 6.2]): μ_1 does not hit $1-\delta$ a.s. iff
- either $\varphi(1-\delta) = +\infty$;
- or $\varphi(1-\delta) < +\infty$ but

$$\int_{1/2}^{1-\delta} m(x)(\varphi(1-\delta) - \varphi(x))dx = +\infty.$$

It suffices to plug b_0 and a_0 into these formulas:

$$\frac{2b_0(z)}{a_0(z)} = \frac{2(\psi(z)(1-z) - \psi(1-z)z)}{2z^2(1-z)^2} = \frac{\psi(z)}{z^2(1-z)} - \frac{\psi(1-z)}{z(1-z)^2} =$$
$$\frac{z(-g(z) + 1/2) - z^2}{z^2(1-z)} - \frac{(1-z)(-g(1-z) + 1/2) - (1-z)^2}{z(1-z)^2} =$$
$$\frac{-g(z)}{z(1-z)} + \frac{g(1-z)}{z(1-z)} + \frac{1-2z}{z(1-z)} = \frac{-g(z)}{z(1-z)} + \varepsilon_1(z),$$

where

$$\varepsilon_1(z) := \frac{g(1-z)}{z(1-z)} + \frac{1-2z}{z(1-z)}.$$

Since the function g is continuous on $[\delta, 1/2]$, there exists a constant $C_1 > 0$ such that for every $z \in [1/2, 1-\delta]$ we have: $|\varepsilon(z)| \le C_1$. Therefore,

$$G(y) := \exp\left(\int_{1/2}^y -\frac{2b_0(z)}{a_0(z)}dz\right) = \exp\left(\int_{1/2}^y \left(\frac{g(z)}{z(1-z)} - \varepsilon(z)\right)dz\right) = e^{F(y)}\varepsilon_2(y),$$

where

$$\varepsilon_2(y) := \exp\left(-\int_{1/2}^{y} \varepsilon_1(z)dz\right), \quad \text{for} \;\; y \in [1/2, 1-\delta).$$

Let $C_2 := \exp((1/2-\delta)C_1)$. Then for every $y \in [1/2, 1-\delta)$ we have:

$$C_2^{-1} \le \varepsilon_2(y) \le C_2.$$

Therefore, the integrals

$$\varphi(1-\delta) = \int_{1/2}^{1-\delta} G(y)dy \;\; \text{and} \;\; \int_{1/2}^{1-\delta} e^{F(y)}dy$$

either both converge or both diverge. If they diverge, then μ_1 does not hit $1-\delta$, and the proof in this case is complete. Otherwise, they both converge, and to finish the proof, we need only to show that

$$\int_{1/2}^{1-\delta} (\varphi(1-\delta) - \varphi(x))m(x)dx < \infty.$$

Indeed,

$$\varphi(1-\delta) - \varphi(x) = \int_{x}^{1-\delta} G(y)dy, \;\; \text{and} \;\; m(x) := \frac{1}{\varphi'(x)a_0(x)} = \frac{1}{G(x)} \cdot \frac{1}{2x^2(1-x)^2}.$$

Therefore, we have:

$$\int_{1/2}^{1-\delta} (\varphi(1-\delta) - \varphi(x))m(x)dx = \int_{1/2}^{1-\delta}\int_{x}^{1-\delta} G(y)dy \cdot \frac{1}{G(x)} \cdot \frac{dx}{2x^2(1-x)^2}. \tag{7}$$

Since $1/(2x^2(1-x)^2)$ is bounded from above for $x \in [1/2, 1-\delta)$, and

$$0 < C_2^{-1} \le \frac{G(x)}{e^{F(x)}} = \varepsilon_2(x) \le C_2 < \infty,$$

the last integral is finite iff the integral

$$\int_{1/2}^{1-\delta}\left(\int_{x}^{1-\delta} e^{F(y)}dy\right)\frac{dx}{e^{F(x)}} \tag{8}$$

is finite. Let us prove that it is indeed finite. Recall that $g(x) \to +\infty$ as $x \uparrow 1-\delta$. There exists $x_0 \in (1/2, 1-\delta)$ such that $g(x) \ge 0$ for $x \in [x_0, 1-\delta)$. Then F increases on $[x_0, 1-\delta)$, and so does e^F. So $e^{F(x)} \ge e^{F(x_0)}$ for $x \in [x_0, 1-\delta)$. Since F is continuous on $[1/2, x_0]$, it is bounded from below on this interval, and so is e^F. Therefore, e^F is bounded from below on $[1/2, 1-\delta)$ by some positive constant $C_3 > 0$. Thus,

$$\int_{1/2}^{1-\delta}\left(\int_{x}^{1-\delta} e^{F(y)}dy\right)\frac{dx}{e^{F(x)}} \le \frac{1}{C_3}\int_{1/2}^{1-\delta}\left(\int_{1/2}^{1-\delta} e^{F(y)}dy\right)dx = \frac{(1-\delta)-1/2}{C_3}\int_{1/2}^{1-\delta} e^{F(y)}dy < \infty. \;\blacksquare$$

Proof of Corollary 2.3. (i) Let $A(z) = 1/(z(1-z))$. Then $C_0 = A(1-\delta)$. Let us show that

$$\int_{1/2}^{1-\delta} \exp\left(\int_{1/2}^{y} A(z)g(z)dz\right)dy = \infty. \tag{9}$$

Since the function A is continuous and increasing on $[1/2, 1-\delta]$, there exists $x_0 \in (1/2, 1-\delta)$ such that for $z \in [x_0, 1-\delta]$ we have: $C_0 - \varepsilon \le A(z) \le C_0$. Since $g(1-\delta) = \infty$, w.l.o.g. we can assume that for $z \in [x_0, 1-\delta)$ we have: $g(z) \ge 0$. To prove (9), it suffices to show that

$$\int_{x_0}^{1-\delta} \exp\left(\int_{1/2}^{y} A(z)g(z)dz\right)dy = \infty. \tag{10}$$

The expression under the outer integral can be rewritten as

$$\exp\left(\int_{x_0}^{y} A(z)g(z)dz\right)\cdot\exp\left(\int_{1/2}^{x_0} A(z)g(z)dz\right).$$

The second multiple does not depend on y. Therefore, to show (10), it suffices to prove that

$$\int_{x_0}^{1-\delta}\exp\left(\int_{x_0}^{y} A(z)g(z)dz\right)dy=\infty.$$

However, since $A(z)\ge C_0-\varepsilon$ and $g(z)\ge 0$ for $z\in[x_0,1-\delta)$, we have:

$$\int_{x_0}^{1-\delta}\exp\left(\int_{x_0}^{y} A(z)g(z)dz\right)dy\ge\int_{x_0}^{1-\delta}\exp\left((C_0-\varepsilon)\int_{1/2}^{y} g(z)dz\right)dy.$$

It follows from (6) that

$$\int_{x_0}^{1-\delta}\exp\left((C_0-\varepsilon)\int_{1/2}^{y} g(z)dz\right)dy=\infty.$$

The proof of (i) is complete. (ii) is proved analogously. Let us show (iii). Assume $\int_{1/2}^{1-\delta} g(z)dz<\infty$. The function A is continuous (therefore, bounded) on $[1/2,1-\delta]$. Therefore, for $y\in[1/2,1-\delta)$ the function

$$F(y):=\int_{1/2}^{y} g(z)A(z)dz\ \text{ is bounded from above}$$

and e^F is bounded (therefore, integrable) on $[1/2,1-\delta)$. ■

4. Proof of Theorem 2.4 and Corollary 2.5

Proof of Theorem 2.4. The proof also uses Feller's test as a main tool, but there are some caveats. Let us first informally present the idea of the proof.

Step 1: Choose a market weight, say μ_1, and find whether it hits 0 or $1-\delta$. Write an equation for μ_1 in the form:

$$d\mu_1(t)=\beta(t)dt+\rho(t)dB(t). \tag{11}$$

Here, $\beta=(\beta(t),t\ge 0)$ and $\rho=(\rho(t),t\ge 0)$ are certain random processes, and $B=(B(t),t\ge 0)$ is a one-dimensional standard Brownian motion.

Step 2: The first caveat is that $\beta(t)$ and $\rho(t)$ are *not* functions of $\mu_1(t)$. In fact, these are functions of the *whole* market weights vector $\mu(t)=(\mu_1(t),\ldots,\mu_n(t))$. In the case of two stocks ($n=2$), we could express the other market weight $\mu_2(t)$ as $1-\mu_1(t)$, so $\beta(t)$ and $\rho(t)$ were functions of only $\mu_1(t)$. This means that μ_1 satisfies an SDE and we could just apply Feller's test in the most straightforward manner.

Here, however, the equation (11) is not an SDE, so μ_1 is not a diffusion process. We find some lower and upper estimates for $\beta(t)$ and $\rho(t)$ of the form:

$$b_1(\mu_1(t))\le\beta(t)\le b_2(\mu_1(t))\ \text{ and }\ \sigma_1(\mu_1(t))\le\rho(t)\le\sigma_2(\mu_1(t)). \tag{12}$$

Step 3: Having found these estimates, one might want to compare μ_1 to solutions of SDEs. However, this approach does not work directly. When we compare two Itô processes, they must have the same diffusion coefficient, like in [KS91, Proposition 5.2.18]. We resolve this difficulty by the following trick: making the diffusion coefficients equal to each other by an appropriate time-change, as in [Haj85, Section 3]. On the general theory of time-change, see [KS91, Section 3.4.B].

In this fashion, we can carry out this comparison. Then we use Feller's test to find whether the solutions of these SDEs hit or do not hit 0 and $1-\delta$. If they hit 0, this contradicts the assumption

that the solutions lie in $(0,\infty)^n$. So we will prove that the solutions do not hit 0. And whether it hits or does not hit $1-\delta$ defines whether this market is diverse or not.

Now let us carry out the proof of Theorem 2 in full detail, following the steps outlined above.

Step 1: Choose one of the market weights, say μ_1. Let us find whether it hits 0 or $1-\delta$. It satisfies the equation

$$d\mu_1(t) = \left[\psi(\mu_1(t)) - \mu_1(t)\sum_{j=1}^n \psi(\mu_j(t))\right]dt + \mu_1(t)dW_1(t) - \mu_1(t)\sum_{j=1}^n \mu_j(t)dW_j(t)\,,$$

which we rewrite as

$$d\mu_1(t) = \beta(t)dt + \rho(t)dB(t).$$

Here $B = (B(t), t \ge 0)$ is a one-dimensional standard Brownian motion, and

$$\beta(t) = \psi(\mu_1(t))(1-\mu_1(t)) - \mu_1(t)\sum_{j=2}^n \psi(\mu_j(t)), \qquad \rho(t) = \sqrt{(\mu_1(t)-\mu_1^2(t))^2 + \mu_1^2(t)\sum_{j=2}^n \mu_j^2(t)}$$

are drift and diffusion terms. As mentioned above, they depend on the whole market weights vector $\mu(t)$, not just on $\mu_1(t)$.

Step 2: Let us find lower and upper estimates for $\beta(t)$ and $\rho(t)$ which depend only on $\mu_1(t)$ and not on other market weights.

(a) First, consider $\rho(t)$. We have:

$$\sum_{j=2}^n \mu_j(t) = 1-\mu_1(t), \quad \text{and} \quad \mu_j(t) > 0 \ \text{ for } \ j = 2,\dots,n.$$

Note that $\sum_{j=2}^n \mu_j(t) = 1-\mu_1(t)$ and $\mu_j(t) > 0$ for $j = 2,\dots,n$. Consider the expression

$$y_1^2 + \ldots + y_m^2, \quad \text{where } \ y_1,\dots,y_m \ge 0 \qquad \text{and} \qquad y_1+\ldots+y_m = a > 0.$$

Its maximal value is achieved when one of the variables y_i is equal to a and all others are 0 (the maximal value is equal to a^2). Its minimal value is achieved when all variables are equal (i.e. equal to a/m), and the minimal value is equal to $m(a/m)^2 = a^2/m$. Therefore,

$$(n-1)^{-1}(1-\mu_1(t))^2 \le \sum_{j=2}^n \mu_j^2(t) \le (1-\mu_1(t))^2.$$

We have the following estimates for $\rho(t)$:

$$\sqrt{(\mu_1(t)-\mu_1^2(t))^2 + (n-1)^{-1}\mu_1^2(t)(1-\mu_1(t))^2} \le \rho(t) \le \sqrt{(\mu_1(t)-\mu_1^2(t))^2 + \mu_1^2(t)(1-\mu_1(t))^2}.$$

Rewrite this as

$$\varkappa\sigma(\mu_1(t)) \le \rho(t) \le \sigma(\mu_1(t)), \tag{13}$$

where we denote

$$\sigma(x) := \sqrt{(x-x^2)^2 + x^2(1-x)^2} = \sqrt{2}x(1-x), \quad \text{and} \quad \varkappa = \left(\frac{1+(n-1)^{-1}}{2}\right)^{1/2}.$$

Note that for $x \in (0, 1-\delta)$ we have:

$$0 < \sigma(x) \le K := \sqrt{2}/4 < \infty, \quad \text{and} \quad 0 < \varkappa < 1. \tag{14}$$

This will be used in the sequel, when we define the time-change.

(b) Let us find similar estimates for the drift $\beta(t)$. Condition (5) implies that the function $\psi(x) := x(1/2 - g(x)) - x^2$ is bounded on $(0, 1/2]$. Also, $\lim_{x\to 1-\delta}\psi(x) = -\infty$, so ψ is bounded

from above on $(0, 1-\delta)$. Therefore, there exists a constant $C_1 > 0$ such that for every $(y_1, \ldots, y_{n-1})$ such that $0 < y_i < 1-\delta,\ i = 1, \ldots, n-1$ and $y_1 + \ldots + y_{n-1} < 1$, we have:

$$\sum_{j=1}^{n-1}\psi(y_j) \le C_1.$$

Therefore, $\sum_{j=2}^{n}\psi(\mu_j(t)) \le C_1$. Thus,

$$\beta(t) \ge b_1(\mu_1(t)),\quad \text{where}\ \ b_1(x) := \psi(x)(1-x) - C_1x. \tag{15}$$

We can get an upper estimate under additional condition $\mu_1(t) > x_0$. Let

$$C_2 := \min\sum_{j=1}^{n-1}\psi(y_j),$$

where the minimum are taken over all $(y_1, \ldots, y_{n-1})$ satisfying $y_j > 0$, for $j = 1, \ldots, n-1$, and $y_1 + \ldots + y_{n-1} \le x_0$. Since $y_1 = \mu_2(t), \ldots,\ y_{n-1} = \mu_n(t)$ satisfy these conditions, we have:

$$\beta(t) \le b_2(\mu_1(t)),\quad \text{where}\ \ b_2(x) := \psi(x)(1-x) - C_2x. \tag{16}$$

Step 3: Now let us do the time-change. Let

$$\Delta(t) = \int_0^t\frac{\rho^2(u)}{\sigma^2(\mu_1(u))}du.$$

This is a strictly increasing, continuously differentiable function with $\Delta(0) = 0$ and $\Delta(\infty) = \infty$, and with $\varkappa^2 \le \Delta'(t) \le 1$ for all $t \ge 0$. Let τ be its inverse function and let $X(t) = \mu_1(\tau(t))$. Then, by Lemma 5.1 (see Appendix) ,

$$dX(s) = \beta(\tau(s))\frac{\rho^2(\tau(s))}{\sigma^2(X(s))}ds + \sigma(X(s))dB_0(s),$$

where $B_0 = (B_0(s), s \ge 0)$ is an $(\mathcal{F}_{\Delta(t)})_{t\ge0}$-Brownian motion.

Our goal is to compare it with solutions of one-dimensional SDEs to find whether X hits or does not hit 0 and $1-\delta$. Note that the range of $\Delta(t)$ is $[0, \infty)$, so X hits a point $a \in \mathbb{R}$ iff μ_1 hits it.

First, let us show X never hits zero. Construct a diffusion Z_0 and compare X with this diffusion. Since $\overline{\lim}_{x\to0}xg(x) < 0$, we have: $\underline{\lim}_{x\to0}\psi(x) > 0$. Therefore, $\underline{\lim}_{x\to0}b_1(x) > 0$. There exist $K_1 > 0$ and $x_1 \in (0, x_0)$ such that for every $x \in (0, x_1)$, we have: $b_1(x) \ge K_1$. If $\mu_1(t) \in (0, x_1)$, then we have:

$$\rho(t) \ge \varkappa\sigma(\mu_1(t))\ \ \text{and}\ \ \beta(t) \ge b_1(\mu_1(t)) > 0.$$

Therefore,

$$\rho(\tau(s)) \ge \varkappa\sigma(X(s)),\ \ \text{and}\ \ \beta(\tau(s)) \ge b_1(X(s)) > 0.$$

Let $Z_0 = (Z_0(s), s \ge 0)$ be the solution of the following SDE:

$$dZ_0(s) = \varkappa^2b_1(Z_0(s))ds + \sigma^2(Z_0(s))dB_0(s),\ \ \text{with initial condition}\ \ Z_0(0) = X(0).$$

By Lemma 5.2 the process X does not hit 0 if the process Z_0 does not hit 0.

It suffices to show that Z_0 does not hit 0. Apply Feller's test: we want to show that

$$\int_0^{x_1}\exp\Bigl(-\int_{x_1}^{y}\frac{2b_1(z)}{\varkappa^2\sigma^2(z)}dz\Bigr)dy = +\infty,\ \ \text{or, in other words,}\ \ \int_0^{x_1}\exp\Bigl(\int_{y}^{x_1}\frac{2b_1(z)}{\varkappa^2\sigma^2(z)}dz\Bigr)dy = +\infty.$$

The function b_1 is bounded from below on $(0, x_1)$. Also, $\sigma_2^2(z) = 2z^2(1-z)^2$. Therefore, for some constant $K_1 > 0$ we have:

$$\int_0^{x_1}\exp\Bigl(K_1\int_y^{x_1}\frac{dz}{z^2(1-z)^2}\Bigr)dy \ge \int_0^{x_1}\exp\Bigl(K_1\int_y^{x_1}\frac{dz}{z^2}\Bigr)dy =$$
$$\int_0^{x_1}\exp\Bigl(K_1(1/y - 1/x_1)\Bigr)dy = e^{-K_1/x_1}\int_0^{x_1}e^{K_1/y}dy = +\infty.$$

The proof of the fact that Z_0 (and, therefore, X, and μ_1) does not hit zero is complete.

Now, let us deal with the other singularity: $1-\delta$.

Since $\lim_{x\to(1-\delta)-} g(x) = +\infty$, we have: $\lim_{x\to(1-\delta)-}\psi(x) = -\infty$. There exists some $x_2 \in (x_0, 1-\delta)$ such that for every $x \in [x_2, 1-\delta)$, we have: $b_2(x) := \psi(x)(1-x) - C_2x < 0$. Therefore, on the event $\{\mu_1(t) \ge x_2\}$ we have:

$$b_1(\mu_1(t)) \le \beta(t) \le b_2(\mu_1(t)) < 0, \ \text{ and } \ \varkappa\sigma(\mu_1(t)) \le \rho(t) \le \sigma(\mu_1(t)).$$

Therefore, on the event $\{X(t) \ge x_2\}$ we have:

$$b_1(X(t)) \le \beta(\tau(t)) \le b_2(X(t)) < 0, \ \text{ and } \ \varkappa\sigma(X(t)) \le \rho(\tau(t)) \le \sigma(X(t)),$$
$$\text{and } \ b_1(X(t)) \le \beta(\tau(t))\frac{\rho^2(\tau(t))}{\sigma^2(X(t))} \le \varkappa^2 b_2(X(t)).$$

We can compare X with the processes $Z_1 = (Z_1(s), s \ge 0)$ and $Z_2 = (Z_2(s), s \ge 0)$ that satisfy the SDEs:

$$dZ_1(s) = b_1(Z_1(t))dt + \sigma(Z_1(s))dW(s), \ \ dZ_2(s) = \varkappa^2 b_2(Z_2(t))dt + \sigma(Z_2(s))dW(s).$$

If Z_1 hits $1-\delta$, then X does. If Z_2 does not hit $1-\delta$, then X does not.

For the processes Z_1 and Z_2, we can again use Feller's test. Consider first the process Z_1. The natural scale for this process is

$$\varphi_1(x) := \int_{x_0}^x \exp\left(\int_{x_0}^y -\frac{2b_1(z)}{\sigma^2(z)}dz\right)dy.$$

We have:

$$-\frac{2b_1(z)}{\sigma^2(z)} = -\frac{2(\psi(z)(1-z) - C_1z)}{2z^2(1-z)^2} = -\frac{\psi(z)}{z^2(1-z)} + \frac{C_1}{z(1-z)^2} =$$
$$-\frac{z(-g(z)+1/2) - z^2}{z(1-z)^2} + \frac{C_1}{z(1-z)^2} = \frac{1}{z(1-z)}g(z) + \varepsilon_3(z),$$

where

$$\varepsilon_3(z) := -\cdot\frac{1/2 - z}{z(1-z)} + \frac{C_1}{z(1-z)^2}.$$

Since the function ε_3 is continuous on $[x_0, 1-\delta]$, it is bounded on this interval: there exists $C_3 > 0$ such that $|\varepsilon_3(z)| \le C_3$ for $z \in [x_0, 1-\delta]$. Therefore,

$$\varphi_1(1-\delta) = \int_{x_0}^{1-\delta}\exp\left(\int_{x_0}^y A_1(z)g(z) + \varepsilon_3(z)\right)dy =$$
$$\int_{x_0}^{1-\delta}\exp\left(\int_{x_0}^y A_1(z)g(z)dz\right)\exp\left(\int_{x_0}^y \varepsilon_3(z)dz\right)dy.$$

For $y \in [x_0, 1-\delta]$, we have:

$$C_4^{-1} \le \exp\left(\int_{x_0}^y \varepsilon_3(z)dz\right) \le C_4, \ \ \text{where } \ C_4 := \exp((1-x_0-\delta)C_3).$$

Therefore,

$$\varphi(1-\delta) < +\infty \ \text{ iff } \ \int_{x_0}^{1-\delta} \exp\left(\int_{1/2}^{y} A_1(z)g(z)dz\right) dy < \infty.$$

If these integrals are indeed finite, then

$$\int_{x_0}^{1-\delta} \frac{\varphi_1(1-\delta)-\varphi_1(x)}{\varphi'(x)\sigma^2(x)} dx < \infty.$$

This is checked just as in the proof of Theorem 1. Thus, if

$$\int_{x_0}^{1-\delta} \exp\left(\int_{x_0}^{y} A_1(z)g(z)dz\right) dy < \infty,$$

then Z_1 hits $1-\delta$ with positive probability, and the market is not diverse. $\varphi_1(1-\delta) = +\infty$ iff

$$\int_{x_0}^{1-\delta} \exp\left(\int_{x_0}^{y} A_1(z)g(z)dz\right) dy = \infty.$$

If $\varphi(1-\delta) < \infty$, then Z_1, and, therefore, X and μ_1 hit $1-\delta$ with positive probability. The proof of (i) is complete. The (ii) part is proved similarly. ■

The proof of Corollary 2.5 is similar to the proof of Corollary 2.3 and is left to the reader.

5. Appendix

Proof of Lemma 2.1. We have

$$dX_i(t) = X_i(t)\left[\left(-g(\mu_i(t)) + \frac{1}{2}\right)dt + dW_i(t)\right], \quad i = 1, \ldots, n$$

and

$$dX(t) = \sum_{i=1}^{n} X_i(t)\left(-g(\mu_i(t)) + \frac{1}{2}\right)dt + \sum_{i=1}^{n} X_i(t)dW_i(t).$$

Apply Ito's formula to $X_i(t)/X(t)$. Let $f(x,y) = x/y$, then

$$f_x = \frac{1}{y},\ f_y = -\frac{x}{y^2},\ f_{xx} = 0,\ f_{xy} = -\frac{1}{y^2},\ f_{yy} = \frac{2x}{y^3}.$$

Therefore, we have:

$$d\mu_i(t) = df(X_i(t), X(t)) = f_x(X_i(t), X(t))dX_i(t) + f_y(X_i(t), X(t))dX(t) +$$
$$\frac{1}{2}f_{xx}(X_i(t), X(t))d < X_i >_t + f_{xy}(X_i(t), X(t))d < X_i, X >_t + \frac{1}{2}f_{yy}(X_i(t), X(t))d < X >_t =$$
$$\frac{1}{X(t)}X_i(t)\left[\left(-g(\mu_i(t)) + \frac{1}{2}\right)dt + dW_i(t)\right] - \frac{X_i(t)}{X^2(t)}\sum_{j=1}^{n} X_j(t)\left[\left(-g(\mu_j(t)) + \frac{1}{2}\right)dt + dW_j(t)\right] -$$
$$\frac{1}{X^2(t)}X_i^2(t)dt + \frac{X_i(t)}{X^3(t)}\sum_{j=1}^{n} X_j^2(t)dt.$$

We can express this in terms of market weights as

$$\mu_i(t)(-g(\mu_i(t)) + 1/2)dt + \mu_i(t)dW_i(t) - \mu_i(t)\sum_{j=1}^{n}\mu_j(t)(-g(\mu_j(t)) + 1/2)dt -$$
$$\mu_i(t)\sum_{j=1}^{n}\mu_j(t)dW_j(t) - \mu_i^2(t)dt + \mu_i(t)\sum_{j=1}^{n}\mu_j^2(t)dt =$$

$$\left[\psi(\mu_i(t)) - \mu_i(t)\sum_{j=1}^n \psi(\mu_j(t))\right]dt + \sum_{j=1}^n (\delta_{ij}\mu_i - \mu_i\mu_j)dW_j(t).\blacksquare$$

Lemma 5.1. *Assume $X = (X_t, t \geq 0)$ is a progressively measurable continuous real-valued process such that*

$$X_t = x + \int_0^t \gamma_t dt + \int_0^t \rho_t dW_t,$$

where $W = (W_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. $\gamma = (\gamma_t)_{t\geq 0}$ and $(\rho_t)_{t\geq 0}$ are progressively measurable processes. Assume that

$$0 < \varkappa \leq \frac{|\rho_t|}{\sigma(X_t)} \leq 1,$$

where $\sigma : \mathbb{R} \to (0, \infty)$ is a real-valued function. Consider the following time-change:

$$\Delta(t) = \int_0^t \frac{\rho_s^2}{\sigma^2(X_s)} ds.$$

This is a strictly increasing function, $\Delta(0) = 0,\ \Delta(\infty) = \infty$. Consider its inverse: $\tau = \Delta^{-1}$. Then the process $Z = (Z_s = X_{\tau(s)}, s \geq 0)$ satisfies the equation

$$dZ_s = \gamma_{\tau(s)} \frac{\rho^2_{\tau(s)}}{\sigma^2(Z_s)} ds + \sigma(Z_s)dB_s,$$

where $B = (B_s)_{s\geq 0}$ is another $(\mathcal{F}_{\Delta(t)})_{t\geq 0}$-Brownian motion.

Proof. Analogous to [Haj85, Section 3]. □

Lemma 5.2. *Assume $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are two progressively measurable continuous real-valued processes which satisfy*

$$dX_t = \beta_t dt + \sigma(X_t)dW_t,\ X_0 = x;\quad dY_t = b(Y_t)dt + \sigma(Y_t)dW_t,\ Y_0 = x.$$

Here, $W = (W_t)_{t\geq 0}$ is a Brownian motion, $\beta = (\beta_t)_{t\geq 0}$ is a progressively measurable process, and $b,\ \sigma : \mathbb{R} \to \mathbb{R}$ are real-valued functions. If $\beta_t \leq b(X_t)$ for $t \geq 0$, then $X_t \leq Y_t$ for $t \geq 0$.

Proof. Follows from [IW89, Lemma 6.1]. □

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